

NON-KOSZUL QUADRATIC GORENSTEIN TORIC RINGS

KAZUNORI MATSUDA

ABSTRACT. Koszulness of Gorenstein quadratic algebras of small socle degree is studied. In this note, we construct non-Koszul Gorenstein quadratic toric ring such that its socle degree is more than 3 by using stable set polytopes.

INTRODUCTION

Let K be a field and $S = K[x_1, \dots, x_n]$ a polynomial ring over K . Let $R = S/I$ be a standard graded K -algebra with respect to the grading $\deg x_i = 1$ for all $1 \leq i \leq n$, where I is a homogeneous ideal of S . Let R_+ denote the homogeneous maximal ideal of R . For an R -module M , we denote $\beta_{ij}^R(M)$ by the (i, j) -th graded betti number of M as an R -module.

The Koszul algebra was originally introduced by Priddy [29]. A standard graded K -algebra R is said to be *Koszul* if the residue field $K = R/R_+$ has a linear R -free resolution as an R -module, that is, $\beta_{ij}^R(K) = 0$ if $i \neq j$. Since $\beta_{2j}^R(K) = 0$ for all $j > 2$, hence Koszul algebras are *quadratic*, where $R = S/I$ is said to be quadratic if I is generated by homogeneous elements of degree 2. Every quadratic complete intersection is Koszul by Tate's theorem [35]. Moreover, $R = S/I$ is Koszul if I has a quadratic Gröbner bases by Fröberg's theorem [10] and the fact that $\beta_{ij}^R(K) \leq \beta_{ij}^{R'}(K)$ for all i, j and for all monomial order $<$ on S , where $R' = S/\text{in}_<(I)$. The notion of Koszul algebra has played an important role in the research on graded K -algebras, and various Koszul-like algebras have been introduced, e.g., universally Koszul [5], strongly Koszul [12], initially Koszul [2], sequentially Koszul [1], etc.

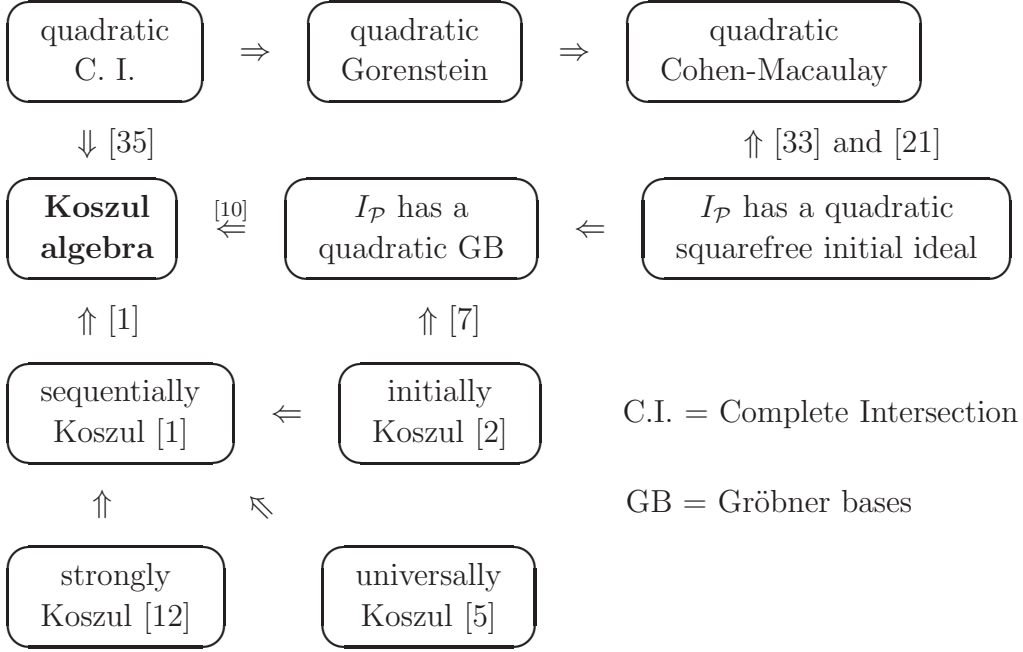
Koszulness of toric rings of integral convex polytopes is studied. Let $\mathcal{P} \subset \mathbb{R}^n$ be an integral convex polytope, i.e., a convex polytope each of whose vertices belongs to \mathbb{Z}^n , and let $\mathcal{P} \cap \mathbb{Z}^n = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$. Assume that $\mathbb{Z}\mathbf{a}_1 + \dots + \mathbb{Z}\mathbf{a}_m = \mathbb{Z}^n$. Let $K[X^{\pm 1}, t] := K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, t]$ be the Laurent polynomial ring in $n + 1$ variables over K . Given an integer vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we put $X^{\mathbf{a}}t = x_1^{a_1} \cdots x_n^{a_n}t \in K[X^{\pm 1}, t]$. The *toric ring* of \mathcal{P} , denoted by $K[\mathcal{P}]$, is the subalgebra of $K[X^{\pm 1}, t]$ generated by $\{X^{\mathbf{a}_1}t, \dots, X^{\mathbf{a}_m}t\}$ over K . Note that $K[\mathcal{P}]$ can be regarded as a standard graded K -algebra by setting $\deg X^{\mathbf{a}_i}t = 1$. The *toric ideal* $\mathcal{I}_{\mathcal{P}}$ is the kernel of a surjective ring homomorphism $\pi : K[Y] = K[y_1, \dots, y_m] \rightarrow K[\mathcal{P}]$ defined

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by $\pi(y_i) = X^{\mathbf{a}_i}t$ for $1 \leq i \leq m$. Then $K[\mathcal{P}] \cong K[Y]/I_{\mathcal{P}}$. It is known that $I_{\mathcal{P}}$ is generated by homogeneous binomials.

Note that the following implications hold:



In addition, it is known the following:

- (1) Conca-De Negri-Rossi posed a conjecture that the defining ideal of a strongly Koszul algebra has a quadratic Gröbner bases [6, Question 13 (1)]. This conjecture is true for the toric ring of edge polytope [16], order polytope [12], stable set polytope [23] and cut polytope [31].
- (2) A squarefree strongly Koszul toric ring is compressed [24, Theorem 2.1], where $K[\mathcal{P}]$ is said to be *compressed* if $\sqrt{\text{in}_{<}(I_{\mathcal{P}})} = \text{in}_{<}(I_{\mathcal{P}})$ for any reverse lexicographic order $<$ on $K[Y]$. In particular, a squarefree strongly Koszul toric ring is quadratic Cohen-Macaulay.
- (3) Many of toric rings associated with integral convex polytopes whose toric ideals has a quadratic Gröbner bases are constructed (e.g., [3], [13], [15], [17], [18], [19]). In other words, many of Koszul toric rings associated with

integral convex polytopes are constructed.

- (4) Quadratic algebra is not always Koszul (see [27, Example 2.1], [30, Example 3]). Note that both of these examples are Cohen-Macaulay but are not Gorenstein.

On the other hand, Koszulness of Gorenstein quadratic algebras is studied. For a standard graded K -algebra $R = \bigoplus_{i \geq 0} R_i$ with $\dim R = d$, we denote by

$$H_R(t) = \sum_{i \geq 0} \dim_K R_i t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

the *Hilbert series* of R , where $h_s \neq 0$, and we say that $h(R) := (h_0, h_1, \dots, h_s)$ is the *h -vector* of R and the index s is the *socle degree* of R . It is known that $h_0 = 1$ and if R is Gorenstein then $h_i = h_{s-i}$ for all $0 \leq i \leq \lfloor s/2 \rfloor$ ([32, Theorem 4.4]). Conca-Rossi-Valla proved that if R is a quadratic Gorenstein with $h(R) = (1, n, 1)$ (in this case $n \geq 2$ since R is quadratic) then R is Koszul [7, Proposition 2.12].

The case for $s = 2$ is also studied. Let R be a quadratic Gorenstein with $h(R) = (1, n, n, 1)$ (in this case $n \geq 3$ since R is quadratic). If $n = 3$, then R is quadratic complete intersection, hence R is Koszul. Conca-Rossi-Valla proved that R is Koszul if $n = 4$ [7, Theorem 6.15] and Caviglia proved that R is Koszul if $n = 5$ in his unpublished master thesis. The case for $n \geq 6$ is still open.

In this note, we focus on (4). In Section 1, we remark about known result of toric rings and toric ideals of stable set polytopes, and construct non-Koszul quadratic Gorenstein toric rings by using stable set polytopes. In Section 2, we present some questions.

Remark 0.1. In this note, we use Macaulay2 [11] to check to be not Koszul. About checking of non-Koszulness by using Macaulay2, see [34, p. 289].

1. STABLE SET POLYTOPE AND NON-KOSZUL QUADRATIC GORENSTEIN TORIC RING

The stable set polytope is an integral convex polytope associated with stable sets of a simple graph.

Let G be a finite simple graph on the vertex set $[n] = \{1, 2, \dots, n\}$ and let $E(G)$ denote the set of edges of G . Recall that a finite graph is *simple* if it possesses no loops or multiple edges. We denote by \overline{G} the complement graph of G .

Given a subset $W \subset [n]$, we define the $(0, 1)$ -vector $\rho(W) = \sum_{i \in W} \mathbf{e}_i \in \mathbb{R}^n$, where \mathbf{e}_i is the i -th unit coordinate vector of \mathbb{R}^n . In particular, $\rho(\emptyset)$ is the origin of \mathbb{R}^n .

A subset $W \subset [n]$ is said to be *stable* if $\{i, j\} \notin E(G)$ for all $i, j \in W$ with $i \neq j$. Note that the empty set and each single-element subset of $[n]$ are stable. Let $S(G)$ denote the set of all stable sets of G . The *stable set polytope* of a simple graph G , denoted by \mathcal{Q}_G , is the convex hull of $\{\rho(W) \mid W \in S(G)\}$. By definition, \mathcal{Q}_G is a $(0, 1)$ -polytope and $K[\mathcal{Q}_G] = K[t \cdot \prod_{i \in W} x_i \mid W \in S(G)] \subset K[x_1, \dots, x_n, t]$. Note that $\dim K[\mathcal{Q}_G] = n + 1$. Let $K[Y] = K[y_W \mid W \in S(G)]$ be the polynomial ring over K . Now we define a surjective ring homomorphism $\pi : K[Y] \rightarrow K[\mathcal{Q}_G]$ by $\pi(y_W) = t \cdot \prod_{i \in W} x_i$ and let $I_{\mathcal{Q}_G} = \ker \pi$.

To state known results of the toric ring $K[\mathcal{Q}_G]$ and the toric ideal $I_{\mathcal{Q}_G}$ of the stable set polytope \mathcal{Q}_G of a simple graph G , we introduce some classes of graphs. About terminologies for the graph theory, see [8].

A *cycle* graph with length n , denoted by C_n , is a connected graph which satisfies $E(C_n) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{1, n\}\}$. An *odd cycle* is a cycle such that its length is odd.

A graph G is said to be *perfect* if the chromatic number of every induced subgraph of G is equal to the size of the largest clique of that subgraph. A graph G is perfect if and only if both G and \overline{G} are $(C_{2n+3}, n \geq 1)$ -free [4].

The *comparability* graph $G(P)$ of a partially ordered set $P = ([n], <_P)$ is the graph such that $V(G(P)) = [n]$ and $\{i, j\} \in E(G(P))$ if and only if $i <_P j$ or $j <_P i$. A graph G is said to be *comparability* if G is the comparability graph of some partially ordered set. Forbidden induced subgraphs of comparability graphs are known (see [22, p.13]).

A graph G is said to be *bipartite* if there exist V_1, V_2 with $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$ such that if $\{i, j\} \in E(G)$ then either $i \in V_1$ and $j \in V_2$ or $i \in V_2$ and $j \in V_1$. It is known that a graph G is bipartite if and only if G is $(C_{2n+1}, n \geq 1)$ -free.

A graph G is said to be *almost bipartite* (see [9, p.87]) if there exists a vertex v such that the induced subgraph $G_{[n] \setminus v}$ is bipartite.

Remark 1.1. It is known that

- (1) Let G be a perfect graph. Then $K[\mathcal{Q}_G]$ is Gorenstein if and only if all maximal cliques of G have the same cardinality [28, Theorem 2.1(b)].
- (2) Let $G(P)$ be the comparability graph of a partially ordered set P . Then $K[\mathcal{Q}_{G(P)}]$ is Koszul since $\mathcal{Q}_{G(P)}$ is equal to the chain polytope of P and the toric ideal of a chain polytope has a squarefree quadratic initial ideal (see [14, Corollary 3.1]).
- (3) If G is almost bipartite, then $K[\mathcal{Q}_G]$ is Koszul since its toric ideal $I_{\mathcal{Q}_G}$ has a squarefree quadratic initial ideal (see [9, Theorem 8.1]).

- (4) Let G be a graph such that \overline{G} is bipartite. Then $K[\mathcal{Q}_G]$ is quadratic if and only of it is Koszul [25, Corollary 3.4].

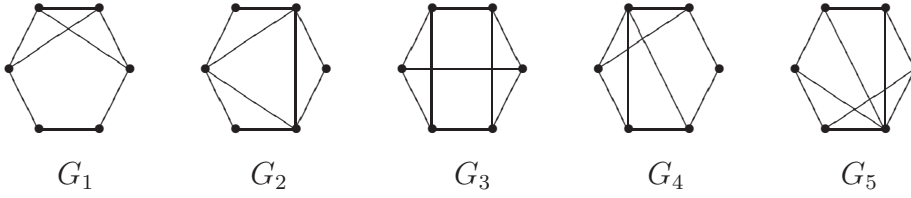
Hence, if $K[\mathcal{Q}_G]$ is quadratic but not Koszul, then G is neither comparability nor almost bipartite, and \overline{G} is not bipartite. From this fact and the classifications of these graphs, we have

Proposition 1.2. *Let G be a graph on $[n]$. If $K[\mathcal{Q}_G]$ is non-Koszul quadratic Gorenstein, then $n \geq 7$.*

Proof. First, we assume that $n \leq 5$. Then G is a comparability graph if G is not C_5 . Since C_5 is almost bipartite, we have that $K[\mathcal{Q}_G]$ is Koszul if $n \leq 5$.

Next, we assume that $n = 6$. If G is not connected, then G is a comparability graph if G is not $C_5 \cup K_1$. Since $C_5 \cup K_1$ is almost bipartite, we have that $K[\mathcal{Q}_{G(P)}]$ is Koszul.

Assume that G is connected. From the classifications of comparability and almost bipartite graphs, G is one of the following (see [23, p.10]):



Then we can see that

- $K[\mathcal{Q}_{G_1}]$ is not Gorenstein since $h(K[\mathcal{Q}_{G_1}]) = (1, 7, 10, 3)$.
- $K[\mathcal{Q}_{G_2}]$ is Koszul since $I_{\mathcal{Q}_{G_2}}$ has a quadratic Gröbner bases.
- $\overline{G_3}$ is C_6 , hence bipartite.
- $K[\mathcal{Q}_{G_4}]$ is not Gorenstein since $h(K[\mathcal{Q}_{G_4}]) = (1, 6, 8, 2)$.
- $K[\mathcal{Q}_{G_5}]$ is Koszul since $I_{\mathcal{Q}_{G_5}} = I_{\mathcal{Q}_{C_5}}$ and $I_{\mathcal{Q}_{C_5}}$ has a quadratic Gröbner bases.

Therefore we have the desired conclusion. \square

For each integer $k \geq 3$, the complement of a odd cycle C_{2k+1} , denoted by $\overline{C_{2k+1}}$, is neither comparability nor almost bipartite. Note that $\overline{C_{2k+1}}$ is not perfect and $S(\overline{C_{2k+1}}) = \{\emptyset, \{1\}, \{2\}, \dots, \{2k+1\}, \{1, 2\}, \{2, 3\}, \dots, \{2k, 2k+1\}, \{1, 2k+1\}\}$.

Let $K[Y] = K[y_\emptyset, y_{\{1\}}, y_{\{2\}}, \dots, y_{\{2k+1\}}, y_{\{1,2\}}, y_{\{2,3\}}, \dots, y_{\{2k,2k+1\}}, y_{\{1,2k+1\}}]$. Now we study the toric ring

$$K[\mathcal{Q}_{\overline{C_{2k+1}}}] \cong \frac{K[Y]}{I_{\mathcal{Q}_{\overline{C_{2k+1}}}}}.$$

Proposition 1.3. We have the following:

- (1) $K[\mathcal{Q}_{\overline{C_{2k+1}}}]$ is quadratic Cohen-Macaulay for all $k \geq 3$.
- (2) $K[\mathcal{Q}_{\overline{C_{2k+1}}}]$ is not Gorenstein for all $k \geq 4$.
- (3) $K[\mathcal{Q}_{\overline{C_7}}]$ is Gorenstein.
- (4) $I_{\mathcal{Q}_{\overline{C_{2k+1}}}}$ possesses no quadratic Gröbner bases for all $k \geq 3$.

Proof. (1) First, by [25, Theorem 2.1], we have that $K[\mathcal{Q}_{\overline{C_{2k+1}}}]$ is normal. Hence $K[\mathcal{Q}_{\overline{C_{2k+1}}}]$ is Cohen-Macaulay. Moreover, by [25, Theorem 3.2], we have that the toric ideal $I_{\mathcal{Q}_{\overline{C_{2k+1}}}}$ is generated by the following $4k+2$ binomials:

- $y_{\{i\}}y_{\{i+1\}} - y_{\emptyset}y_{\{i,i+1\}} \quad (1 \leq i \leq 2k);$
- $y_{\{1\}}y_{\{2k+1\}} - y_{\emptyset}y_{\{1,2k+1\}};$
- $y_{\{i\}}y_{\{i+1,i+2\}} - y_{\{i+2\}}y_{\{i,i+1\}} \quad (1 \leq i \leq 2k-1);$
- $y_{\{2k\}}y_{\{1,2k+1\}} - y_{\{1\}}y_{\{2k,2k+1\}}, y_{\{2k+1\}}y_{\{1,2\}} - y_{\{2\}}y_{\{1,2k+1\}}.$

Hence $K[\mathcal{Q}_{\overline{C_{2k+1}}}]$ is quadratic. Therefore $K[\mathcal{Q}_{\overline{C_{2k+1}}}]$ is quadratic Cohen-Macaulay.

- (2) By (1), $K[\mathcal{Q}_{\overline{C_{2k+1}}}] \cong K[Y]/I_{\mathcal{Q}_{\overline{C_{2k+1}}}}$ is Cohen-Macaulay with $\dim K[\mathcal{Q}_{\overline{C_{2k+1}}}] = 2k+2$. We note that $\mathbf{y} = y_{\emptyset}, y_{\{1\}} - y_{\{2,3\}}, y_{\{2\}} - y_{\{3,4\}}, \dots, y_{\{2k-1\}} - y_{\{2k,2k+1\}}, y_{\{2k\}} - y_{\{1,2k+1\}}, y_{\{2k+1\}} - y_{\{1,2\}}$ is a regular sequence of $K[Y]/I_{\mathcal{Q}_{\overline{C_{2k+1}}}}$. Then we have that

$$\frac{K[Y]}{I_{\mathcal{Q}_{\overline{C_{2k+1}}}} + (\mathbf{y})} \cong \frac{K[y_{\{1\}}, y_{\{2\}}, \dots, y_{\{2k+1\}}]}{I_{2k+1}}$$

is a artinian quadratic Cohen-Macaulay ring, where I_{2k+1} is generated by the followings:

- $y_{\{i\}}y_{\{i+1\}} \quad (1 \leq i \leq 2k);$
- $y_{\{1\}}y_{\{2k+1\}};$
- $y_{\{i\}}^2 - y_{\{i-1\}}y_{\{i+2\}} \quad (2 \leq i \leq 2k-1);$
- $y_{\{1\}}^2 - y_{\{3\}}y_{\{2k+1\}}, y_{\{2k\}}^2 - y_{\{1\}}y_{\{2k-1\}}, y_{\{2k+1\}}^2 - y_{\{2\}}y_{\{2k\}}.$

Assume $k \geq 4$. Then both $y_{\{2k+1\}}^2 \cdot \prod_{i=1}^{k-1} y_{\{2i\}}$ and

$$\begin{cases} \prod_{i=1}^{\frac{2k+1}{3}} y_{\{3i\}} & (k \equiv 1 \pmod{3}), \\ y_{\{2k+1\}} \cdot \prod_{i=1}^{\frac{2k-1}{3}} y_{\{3i\}} & (k \equiv 2 \pmod{3}), \\ y_{\{2k\}}^2 \cdot \prod_{i=1}^{\frac{2k-3}{3}} y_{\{3i\}} & (k \equiv 0 \pmod{3}), \end{cases}$$

are socle elements of $K[y_{\{1\}}, y_{\{2\}}, \dots, y_{\{2k+1\}}]/I_{2k+1}$, hence it is not Gorenstein. Therefore $K[\mathcal{Q}_{\overline{C_{2k+1}}}]$ is not Gorenstein for all $k \geq 4$.

(3) By the proof of (2), we have

$$\frac{K[Y]}{I_{\mathcal{Q}_{\overline{C_7}}} + (\mathbf{y})} \cong \frac{K[y_{\{1\}}, y_{\{2\}}, \dots, y_{\{7\}}]}{I_7}.$$

Let $<_{\text{rev}}$ be the reverse lexicographic order on $K[y_{\{1\}}, y_{\{2\}}, \dots, y_{\{7\}}]$ induced by the ordering $y_{\{1\}} < y_{\{2\}} < \dots < y_{\{7\}}$. Then the initial ideal $\text{in}_{<_{\text{rev}}}(I_7)$ is generated by the following monomials:

$$(y_{\{1\}}y_{\{2\}}, y_{\{2\}}y_{\{3\}}, y_{\{3\}}y_{\{4\}}, y_{\{4\}}y_{\{5\}}, y_{\{5\}}y_{\{6\}}, y_{\{6\}}y_{\{7\}}, y_{\{1\}}y_{\{7\}}, y_{\{1\}}^3, y_{\{2\}}^2, y_{\{3\}}^2, y_{\{4\}}^2, y_{\{5\}}^2, y_{\{6\}}^2, y_{\{7\}}^2, y_{\{3\}}y_{\{7\}}, y_{\{1\}}^2y_{\{4\}}, y_{\{1\}}^2y_{\{6\}}, y_{\{2\}}y_{\{5\}}y_{\{7\}}).$$

From this, we can compute that the Hilbert series of $\frac{K[y_{\{1\}}, y_{\{2\}}, \dots, y_{\{7\}}]}{\text{in}_{<_{\text{rev}}}(I_7)}$ is $1 + 7t + 14t^2 + 7t^3 + t^4$. Hence $h(K[\mathcal{Q}_{\overline{C_7}}]) = (1, 7, 14, 7, 1)$, therefore it is Gorenstein.

(4) Assume that there exists a monomial order $<$ on $K[Y]$ such that the Gröbner bases of $I_{\mathcal{Q}_{\overline{C_{2k+1}}}}$ with respect to $<$ is quadratic.

We may assume that $y_{\{1\}}y_{\{2,3\}} < y_{\{3\}}y_{\{1,2\}}$. Then $y_{\{3\}}y_{\{4,5\}} < y_{\{5\}}y_{\{3,4\}}$ since $y_{\{5\}}y_{\{1,2\}}y_{\{3,4\}} - y_{\{1\}}y_{\{2,3\}}y_{\{4,5\}} \in I_{\mathcal{Q}_{\overline{C_{2k+1}}}}$ and its initial monomial is $y_{\{5\}}y_{\{1,2\}}y_{\{3,4\}}$. Since $y_{\{7\}}y_{\{3,4\}}y_{\{5,6\}} - y_{\{3\}}y_{\{4,5\}}y_{\{6,7\}} \in I_{\mathcal{Q}_{\overline{C_{2k+1}}}}$ and its initial monomial is $y_{\{7\}}y_{\{3,4\}}y_{\{5,6\}}$, we have $y_{\{5\}}y_{\{6,7\}} < y_{\{7\}}y_{\{5,6\}}$. By repeating this argument, we have

$$\begin{aligned} y_{\{1\}}y_{\{2,3\}} &< y_{\{3\}}y_{\{1,2\}}, \\ y_{\{3\}}y_{\{4,5\}} &< y_{\{5\}}y_{\{3,4\}}, \\ &\vdots \\ y_{\{2k-1\}}y_{\{2k,2k+1\}} &< y_{\{2k+1\}}y_{\{2k-1,2k\}}, \\ y_{\{2k+1\}}y_{\{1,2\}} &< y_{\{2\}}y_{\{1,2k+1\}}, \\ y_{\{2\}}y_{\{3,4\}} &< y_{\{4\}}y_{\{2,3\}}, \\ y_{\{4\}}y_{\{5,6\}} &< y_{\{6\}}y_{\{4,5\}}, \\ &\vdots \\ y_{\{2k-2\}}y_{\{2k-1,2k\}} &< y_{\{2k\}}y_{\{2k-2,2k-1\}}, \\ y_{\{2k\}}y_{\{1,2k+1\}} &< y_{\{1\}}y_{\{2k,2k+1\}}. \end{aligned}$$

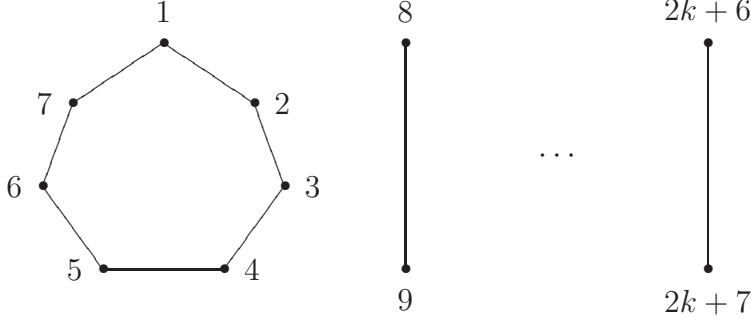
These inequalities induce a contradiction. Hence we have the desired conclusion. \square

We can check that $R = K[\mathcal{Q}_{\overline{C_7}}]$ is not Koszul to check $\beta_{34}^R(K) = 1 \neq 0$ by using Macaulay2. Hence we have

Corollary 1.4. The toric ring $K[\mathcal{Q}_{\overline{C_7}}]$ is non-Koszul quadratic Gorenstein.

We can construct an infinite family of non-Koszul quadratic Gorenstein toric rings by using stable set polytopes.

Proposition 1.5. Let $k \geq 1$ be an integer. Let G be a graph on $[2k + 7]$ such that $\overline{G} = C_7 \cup K_2 \cup \cdots \cup K_2$ and the labeling of vertices is as follows:



Then we have

- (1) $K[\mathcal{Q}_G]$ is quadraic Gorenstein such that

$$H_{K[\mathcal{Q}_G]}(t) = (1 + 7t + 14t^2 + 7t^3 + t^4)(1 + t)^k / (1 - t)^{2k+8}.$$

- (2) $K[\mathcal{Q}_G]$ is not Koszul.

Proof. (1) By [25, Theorem 3.2], we have that the toric ideal $I_{\mathcal{Q}_G}$ is generated by the following binomials:

- $y_{\{i\}}y_{\{i+1\}} - y_{\emptyset}y_{\{i,i+1\}} \quad (1 \leq i \leq 6);$
- $y_{\{1\}}y_{\{7\}} - y_{\emptyset}y_{\{1,7\}};$
- $y_{\{i\}}y_{\{i+1,i+2\}} - y_{\{i+2\}}y_{\{i,i+1\}} \quad (1 \leq i \leq 5);$
- $y_{\{6\}}y_{\{1,7\}} - y_{\{1\}}y_{\{6,7\}}, y_{\{7\}}y_{\{1,2\}} - y_{\{2\}}y_{\{1,7\}};$
- $y_{\{2i\}}y_{\{2i+1\}} - y_{\emptyset}y_{\{2i,2i+1\}} \quad (4 \leq i \leq k + 3).$

Let $K[Y] = K[y_W \mid W \in S(G)]$. Then $K[\mathcal{Q}_G] \cong K[Y]/I_{\mathcal{Q}_G}$. Note that $\mathbf{y} = y_{\emptyset}, y_{\{1\}} - y_{\{2,3\}}, y_{\{2\}} - y_{\{3,4\}}, \dots, y_{\{5\}} - y_{\{6,7\}}, y_{\{6\}} - y_{\{1,7\}}, y_{\{7\}} - y_{\{1,2\}}, y_{\{8\}} - y_{\{9\}}, \dots, y_{\{2k+6\}} - y_{\{2k+7\}}, y_{\{8,9\}}, \dots, y_{\{2k+6,2k+7\}}$ is a regular sequence of $K[Y]/I_{\mathcal{Q}_G}$. Hence we have

$$\frac{K[Y]}{I_{\mathcal{Q}_G} + (\mathbf{y})} \cong \frac{K[y_{\{1\}}, y_{\{2\}}, \dots, y_{\{7\}}]}{I_7} \otimes_K \frac{K[y_{\{2i\}} \mid 4 \leq i \leq k + 3]}{(y_{\{2i\}}^2 \mid 4 \leq i \leq k + 3)}.$$

Thus the Hilbert series of $K[Y]/I_{\mathcal{Q}_G} + (\mathbf{y})$ is $(1 + 7t + 14t^2 + 7t^3 + t^4)(1 + t)^k$. Therefore we have the desired conclusion.

- (2) $K[\mathcal{Q}_{\overline{C_7}}]$ is a combinatorial pure subring (see [26]) of $K[\mathcal{Q}_G]$. Since $K[\mathcal{Q}_{\overline{C_7}}]$ is not Koszul, hence $K[\mathcal{Q}_G]$ is not Koszul by [26, Proposition 1.3]. \square

2. QUESTIONS

As the end of this note, we present some questions.

Recall that the h -vector of $K[\mathcal{Q}_{\overline{C_7}}]$ is $(1, 7, 14, 7, 1)$. Hence the following question is interesting.

Question 2.1. *Does exist a non-Koszul quadratic Gorenstein algebra R such that $h(R) = (1, n_1, n_2, n_1, 1)$ and $n_1 \leq 6$?*

Note that, in this case $n_1 \geq 4$ since R is quadratic. Since $n_1 = \text{embdim } R - \dim R$ and $\text{embdim } K[\mathcal{Q}_G] = \#S(G) = 1 + n + \#\{W \in S(G) \mid \#W \geq 2\}$ and $\dim K[\mathcal{Q}_G] = n + 1$, if $\text{embdim } K[\mathcal{Q}_G] - \dim K[\mathcal{Q}_G] \leq 6$, then $\#\{W \in S(G) \mid \#W \geq 2\} \leq 6$. In particular, we have $\alpha(G) = 2$, where $\alpha(G) := \max\{\#W \mid W \in S(G)\}$ is the *stability number* of G . Since if G is perfect graph with $\alpha(G) = 2$ then \overline{G} is bipartite, In this case G is not perfect.

Let G be a graph on $[n]$ and with $E(G)$ its edge set. The *edge ring* of G , denoted by $K[G]$, is defined by

$$K[G] := K[x_i x_j \mid \{i, j\} \in E(G)] \subset K[x_1, \dots, x_n].$$

The second question is

Question 2.2. *Does exist a graph G such that the edge ring $K[G]$ is non-Koszul quadratic Gorenstein ?*

In [27, Theorem 1.2], a criterion for the edge ring $K[G]$ of G to be quadratic is given. Moreover, in [20], a class of graphs with the property that the toric ideal I_G of the edge ring $K[G]$ of G is quadratic but I_G possesses no quadratic Gröbner bases is studied. A graph G is said to be *(*)-minimal* if G satisfies the above property and every induced subgraph $H \subsetneq G$ does not satisfy the property. By the computation by using Macaulay2, we have that if G is *(*)-minimal* and the edge ring $K[G]$ is non-Koszul quadratic Gorenstein, then $n \geq 9$.

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(Kazunori Matsuda) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL
OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871,
JAPAN

E-mail address: `kaz-matsuda@ist.osaka-u.ac.jp`